九十四學年第一學期 PHYS2310 電磁學 期末考試題(共兩頁)

[Griffiths Ch. 4-6]　2006/01/10,　10:10–12:00am, 教師：張存績

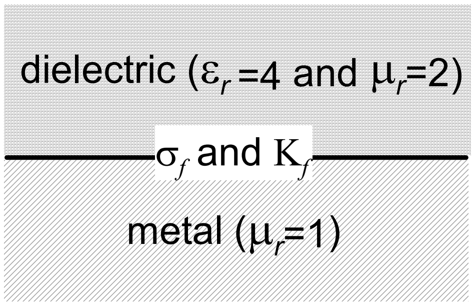
※寫上姓名，學號及班別。並依題號順序每頁答一題。

1. (4%, 4%, 4%, 4%)
   (a) Write down and prove the boundary conditions of **D** using the divergence theorem and Stokes' theorem.
   (b) Write down and prove the boundary conditions of **H** using the divergence theorem and Stokes' theorem.

   At the interface between a metal and a linear dielectric (whose parameters are shown in the figure),

   (c) write down $\mathbf{E}^{\perp}_{\text{above}}$, $\mathbf{E}^{\perp}_{\text{below}}$, $\mathbf{E}^{\parallel}_{\text{above}}$, and $\mathbf{E}^{\parallel}_{\text{below}}$ in terms of surface charge σ_f;
   (d) write down $\mathbf{B}^{\perp}_{\text{above}}$, $\mathbf{B}^{\perp}_{\text{below}}$, $\mathbf{B}^{\parallel}_{\text{above}}$, and $\mathbf{B}^{\parallel}_{\text{below}}$ in terms of surface current $\mathbf{K}_f$.

   [Assume that the magnetic field is generated by the surface current.].

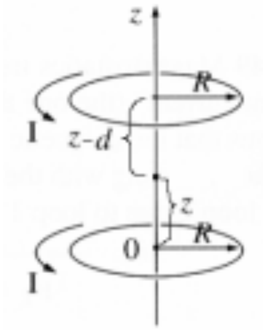


2. (6%, 6%, 5%)
   (a) Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I.

   Helmholtz coil: a convenient way of producing uniform field.

   (b) Find the magnetic field (B) along the z axis as a function of z.
   (c) Show $\partial B/\partial z$ is zero at the point midway between them.

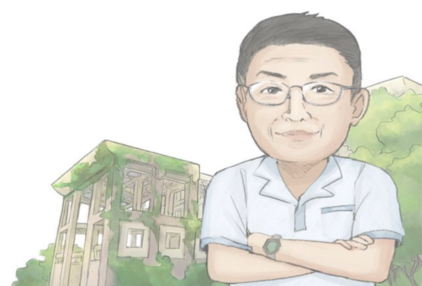
   [Hint: Use the coordinate defined in the figure.]



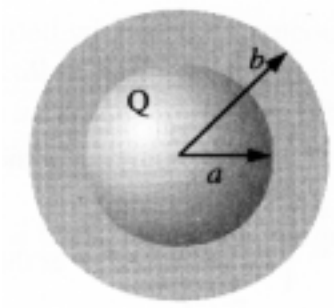
3. (3%, 3%, 5%, 5%) If $\mathbf{J}_f = 0$ everywhere, the curl of **H** vanishes, and we can express **H** as the gradient of a scalar potential W: $\mathbf{H} = -\nabla W$.
   (a) Show that $\nabla^2 W = (\nabla \cdot \mathbf{M})$. [Hint: use the divergence of **H**].

   Consider a uniformly magnetized sphere ($\mathbf{M} = M\hat{\mathbf{z}}$) of radius R.

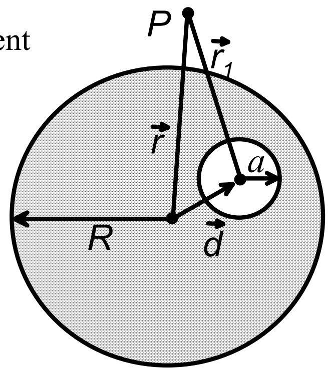
   (b) Write down W in the regions $r < R$ and $r > R$ where $\nabla \cdot \mathbf{M} = 0$ using Legendre polynomial.
   (c) Write down the boundary conditions, and determine the coefficient.
   (d) Find the magnetic field **B** outside of the magnetized sphere.



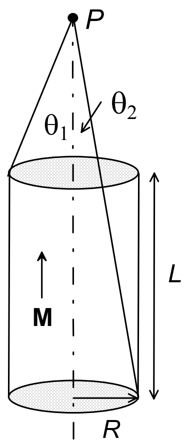
4. (3%, 4%, 4%, 4%, 4%) A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material with susceptibility χ_e.
   (a) Find the electric displacement **D** in the regions $r < a$, $a \le r \le b$, and $r > b$;
   (b) Find the polarization **P** in the regions $a \le r \le b$ and $r > b$;
   (c) Find the bound volume charge ρ_b and the bound surface charge σ_b in the dielectric material;
   (d) Find the electric field **E** in the regions $r < a$, $a \le r \le b$, and $r > b$;
   (e) Find the potential on the surface of the metal sphere relative to infinity.

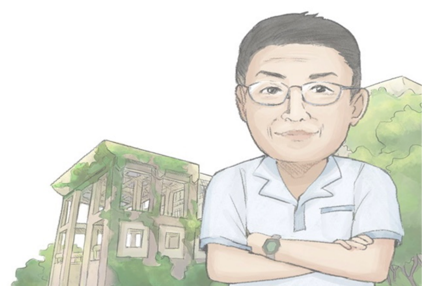


5. (8%, 8%) A long cylindrical conductor of radius R carrying a uniform current density J_0 in the z direction (pointing outward) has a cylindrical hole of radius a along its entire length, as shown in the figure. The center of this hole is offset from the center of the conductor by a distance d.
   (a) Find the magnetic field **B** outside the conductor.
   (b) Find the magnetic field **B** at any point inside the hole.
   [Hint: Express **B** in the vector form and use the principle of superposition.]



6. (8%, 8%) A bar magnet of radius R and length L is magnetized with a uniform magnetization **M** in the z axis as shown in the figure.
   (a) Find the bound volume current $\mathbf{J}_b$ inside the magnet and the bound surface currents $\mathbf{K}_b$ on both ends and the cylindrical surface.
   (b) Find the magnetic field along the z axis. Use the technique similar to that of the solenoid and express you answer in terms of θ_1 and θ_2.





1.

(a) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{enc}$, $(\mathbf{D}^{\perp}_{above} - \mathbf{D}^{\perp}_{below})a = \sigma_f a \;\;\Rightarrow \mathbf{D}^{\perp}_{above} - \mathbf{D}^{\perp}_{below} = \sigma_f$

$\oint \mathbf{E}\cdot d\ell = 0$, $(\mathbf{D}^{\parallel}_{above} - \mathbf{D}^{\parallel}_{below})\ell = (\mathbf{P}^{\parallel}_{above} - \mathbf{P}^{\parallel}_{below})\ell \;\;\Rightarrow \mathbf{D}^{\parallel}_{above} - \mathbf{D}^{\parallel}_{below} = \mathbf{P}^{\parallel}_{above} - \mathbf{P}^{\parallel}_{below}$

(b) $\oint \mathbf{B}\cdot d\mathbf{a} = 0$, $(\mathbf{H}^{\perp}_{above} - \mathbf{H}^{\perp}_{below})a = (\mathbf{M}^{\perp}_{above} - \mathbf{M}^{\perp}_{below})a \;\;\Rightarrow \mathbf{H}^{\perp}_{above} - \mathbf{H}^{\perp}_{below} = \mathbf{M}^{\perp}_{above} - \mathbf{M}^{\perp}_{below}$

$\oint \mathbf{H}\cdot d\ell = I_{enc}$, $(\mathbf{H}^{\parallel}_{above} - \mathbf{H}^{\parallel}_{below})\ell = \mathbf{K}_f \ell \;\;\Rightarrow \mathbf{H}^{\parallel}_{above} - \mathbf{H}^{\parallel}_{below} = \mathbf{K}_f$

(c) $\mathbf{D}^{\perp}_{below} = 0, \quad \mathbf{D}^{\perp}_{above} = \sigma_f = \varepsilon \mathbf{E}^{\perp}_{above} \;\;\Rightarrow\;\; \mathbf{E}^{\perp}_{below} = 0,\; \mathbf{E}^{\perp}_{above} = \sigma_f / 4\varepsilon_0$

$\mathbf{D}^{\parallel}_{below} = 0,\; \mathbf{D}^{\parallel}_{above} = 0 \;\;\Rightarrow\;\; \mathbf{E}^{\parallel}_{below} = 0,\; \mathbf{E}^{\parallel}_{above} = 0$

(d) $\mathbf{H}^{\perp}_{above} = \mathbf{H}^{\perp}_{below} = 0 \;\;\Rightarrow\;\; \mathbf{B}^{\perp}_{above} = 0, \quad \mathbf{B}^{\perp}_{below} = 0$

$\mathbf{H}^{\parallel}_{above} = -\mathbf{H}^{\parallel}_{below} = \mathbf{K}/2 \;\;\Rightarrow\;\; \mathbf{B}^{\parallel}_{above} = \mu \mathbf{H}^{\parallel}_{above} = \mu_0 \mathbf{K}_f, \quad \mathbf{B}^{\parallel}_{below} = \mu \mathbf{H}^{\parallel}_{below} = -\mu_0 \mathbf{K}_f/2$

2.

(a) $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \dfrac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$. Choose a cylindrical coordinate (s, ϕ, z).

In the diagram only the z-component of $(d\mathbf{l}'\times\hat{\mathscr{r}})$ survives.

z-component of $(d\mathbf{l}'\times\hat{\mathscr{r}}) = dl'\cos\theta = R\cos\theta d\phi$

$\dfrac{1}{\mathscr{r}^2} = \dfrac{1}{(R^2+z^2)}$ and $\sin\theta = \dfrac{R}{(R^2+z^2)^{1/2}}$

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R}{(R^2+z^2)(R^2+z^2)^{1/2}} R d\phi = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

(b) $B_z(z) = B_{up}(z-d) + B_{down}(z)$

$B_{down}(z) = \dfrac{\mu_0 I}{2}\dfrac{R^2}{(R^2+z^2)^{3/2}}$ and $B_{up}(d) = \dfrac{\mu_0 I}{2}\dfrac{R^2}{(R^2+(z-d)^2)^{3/2}}$

$$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} + \frac{\mu_0 I}{2}\frac{R^2}{(R^2+(z-d)^2)^{3/2}}$$

(c) $\dfrac{\partial B(z)}{\partial z} = -\dfrac{\mu_0 I R^2}{2}\dfrac{3}{2}[2z(R^2+z^2)^{5/2} + 2(z-d)(R^2+(z-d)^2)^{5/2}], \quad \dfrac{\partial B(d/2)}{\partial z} = 0$

$\dfrac{\partial B(z)}{\partial z} = -\dfrac{\mu_0 I R^2}{2}\dfrac{3}{2}[2z(R^2+z^2)^{5/2} + 2(z-d)(R^2+(z-d)^2)^{5/2}], \quad \dfrac{\partial B(d/2)}{\partial z} = 0$

(d) Determine d such that $\partial^2 B/\partial z^2 = 0$ at the midpoint, and find the resulting magnetic field at the center.

$$\frac{\partial^2 B(z)}{\partial z^2} = -\frac{3\mu_0 I R^2}{2}\frac{\partial}{\partial z}[z(R^2+z^2)^{5/2} + (z-d)(R^2+(z-d)^2)^{5/2}]$$

$$= -\frac{3\mu_0 I R^2}{2}[(R^2+z^2)^{-5/2} - 5z^2(R^2+z^2)^{-7/2} + (R^2+(z-d)^2)^{-5/2} - 5(z-d)^2(R^2+(z-d)^2)^{-7/2}]$$

$$\frac{\partial^2 B(d/2)}{\partial z^2} = -\frac{3\mu_0 I R^2}{2}(R^2+(\frac{d}{2})^2)^{-7/2}\left(2(R^2+(\frac{d}{2})^2)-5(\frac{d}{2})^2\right)$$

$$\frac{\partial^2 B(d/2)}{\partial z^2} = 0, \text{ i.e. } \left((R^2+(\frac{d}{2})^2) = 5(\frac{d}{2})^2\right) \Rightarrow d = R$$

3.

(a) $\nabla\cdot\mathbf{H} = \nabla\cdot(\frac{\mathbf{B}}{\mu_0}-\mathbf{M}) = \frac{1}{\mu_0}\nabla\cdot\mathbf{B}-\nabla\cdot\mathbf{M} = -\nabla^2 W \quad \Rightarrow \nabla^2 W = \nabla\cdot\mathbf{M}$

(b) $\nabla^2 W = 0$ for $r < R$ and $r > R$ $\begin{cases} W_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \\ W_{out}(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) \end{cases}$

(c) $$\begin{cases} W_{in}(R,\theta) = W_{out}(R,\theta) \\ -\left.\frac{\partial W_{in}(r,\theta)}{\partial r}\right|_{r=R} + \left.\frac{\partial W_{out}(r,\theta)}{\partial r}\right|_{r=R} = M\cos(\theta) \end{cases}$$

$$\begin{cases} A_\ell R^\ell = B_\ell R^{-(\ell+1)} \\ \sum_{\ell=0}^{\infty}(-\ell A_\ell R^{\ell-1}+(\ell+1)B_\ell R^{-(\ell+2)})P_\ell(\cos\theta) = M\cos(\theta) \end{cases}$$

$$\ell = 1, \begin{cases} A_1 R = B_1 R^{-2} \\ (-A_1+2B_1R^{-3}) = M \end{cases} \Rightarrow \begin{cases} A_1 = M/3 \\ B_1 = MR^3/3 \end{cases}$$

$\ell \neq 1, \ A_\ell = B_\ell = 0$

(d) $W_{out}(r,\theta) = \frac{MR^3}{3}\frac{1}{r^2}\cos\theta$

$$\mathbf{B}_{out}(r,\theta) = -\mu_0\vec{\nabla}W_{out} = -\mu_0\frac{MR^3}{3}(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{\partial}{r\partial\theta})\frac{1}{r^2}\cos\theta = \mu_0\frac{MR^3}{3}(\hat{\mathbf{r}}\frac{2\cos\theta}{r^3}+\hat{\boldsymbol{\theta}}\frac{\sin\theta}{r^3})$$

$$W_{in} = \frac{M}{3}r\cos\theta = \frac{M}{3}z \quad \Rightarrow \quad \mathbf{B}_{in}(r,\theta) = \mu_0(-\vec{\nabla}W_{in}+\mathbf{M}) = \frac{2}{3}\mu_0 M\hat{\mathbf{z}} \quad \text{(reference)}$$

4.

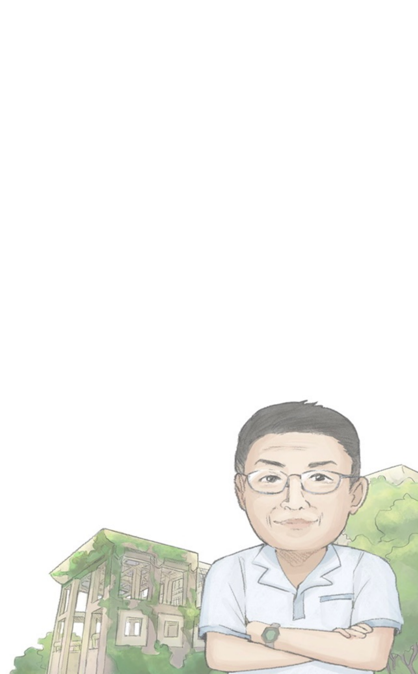
(a) $\mathbf{D} = \frac{Q_{f,enc}}{4\pi r^2}\hat{\mathbf{r}} \quad \Rightarrow \quad \mathbf{D}(r<a) = 0, \ \mathbf{D}(a\le r\le b) = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \ \mathbf{D}(r>b) = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}$

(b) $\mathbf{P} = \chi_e\varepsilon_0\mathbf{E} = \frac{\chi_e}{(1+\chi_e)}\mathbf{D} \quad \Rightarrow \quad \mathbf{P}(a\le r\le b) = \frac{\chi_e}{(1+\chi_e)}\frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \ \mathbf{P}(r>b) = 0$

(c) $\rho_b(a\le r\le b) = -\nabla\cdot\mathbf{P} = \frac{\chi_e}{(1+\chi_e)}\frac{2Q}{4\pi r^3}$

$$\sigma_b(r=a) = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{\chi_e}{(1+\chi_e)}\frac{Q}{4\pi a^2}\hat{\mathbf{r}}\cdot(-\hat{\mathbf{r}}) = -\frac{\chi_e}{(1+\chi_e)}\frac{Q}{4\pi a^2}$$

$$\sigma_b(r=b) = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{\chi_e}{(1+\chi_e)}\frac{Q}{4\pi b^2}\hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}) = \frac{\chi_e}{(1+\chi_e)}\frac{Q}{4\pi b^2}$$



(d) $\mathbf{E}=\dfrac{\mathbf{D}}{(1+\chi_e)\varepsilon_0} \quad \Rightarrow \ \mathbf{E}(a\le r\le b)=\dfrac{Q}{4\pi r^2(1+\chi_e)\varepsilon_0}\hat{\mathbf{r}},\ \ \mathbf{E}(r>b)=\dfrac{Q}{4\pi r^2\varepsilon_0}\hat{\mathbf{r}}$

(e) $V(r=a)=-\int_\infty^b \mathbf{E}\cdot d\ell-\int_b^a \mathbf{E}\cdot d\ell=\dfrac{Q}{4\pi\varepsilon_0}(\dfrac{1}{b})+\ \dfrac{Q}{4\pi(1+\chi_e)\varepsilon_0}(\dfrac{1}{a}-\dfrac{1}{b})$

(f) Find the electrostatic energy of this configuration.

5.

(a) Outside the conductor: Use the principle of superposition $\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2$

Consider a conductor of radius R carrying a current density $\mathbf{J}=J_0\hat{\mathbf{z}}$,

and another conductor of radius a carrying a current density $\mathbf{J}=-J_0\hat{\mathbf{z}}$.

$$\mathbf{B}_1=\frac{\mu_0 J_0 R^2}{2r}\hat{\mathbf{z}}\times\hat{\mathbf{r}},\ \text{ and }\ \mathbf{B}_2=-\frac{\mu_0 J_0 a^2}{2r_1}\hat{\mathbf{z}}\times\hat{\mathbf{r}}_1=-\frac{\mu_0 J_0 a^2}{2r_1}\hat{\mathbf{z}}\times(\hat{\mathbf{r}}-\hat{\mathbf{d}})$$

$$\mathbf{B}=\frac{\mu_0 J_0}{2}\left\{[\frac{R^2}{r}-\frac{a^2}{r_1}]\hat{\mathbf{z}}\times\hat{\mathbf{r}}+\frac{a^2}{r_1}\hat{\mathbf{z}}\times\hat{\mathbf{d}}\right\}$$

(b) Inside the hole

$$\mathbf{B}_1=\frac{\mu_0 J_0 r}{2}\hat{\mathbf{z}}\times\hat{\mathbf{r}}=\frac{\mu_0 J_0}{2}\hat{\mathbf{z}}\times\mathbf{r},$$

$$\mathbf{B}_2=-\frac{\mu_0 J_0 r_1}{2}\hat{\mathbf{z}}\times\hat{\mathbf{r}}_1=-\frac{\mu_0 J_0}{2}\hat{\mathbf{z}}\times\mathbf{r}_1=\frac{\mu_0 J_0}{2}\hat{\mathbf{z}}\times(\mathbf{d}-\mathbf{r})$$

$$\mathbf{B}=\frac{\mu_0 J_0}{2}\hat{\mathbf{z}}\times\mathbf{d}$$

6.

(a) $\mathbf{J}_b=\nabla\times\mathbf{M}=0$ inside the magnet.

$\mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=M\hat{\phi}$ on the surface of the cylinder.

$\mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=M\hat{\mathbf{z}}\times\hat{\mathbf{z}}=0$ on both ends.

(b) $\mathbf{B}(\mathbf{r})=\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\mathbf{K}\times\hat{\mathscr{r}}}{\mathscr{r}^2}da=\frac{\mu_0}{4\pi}\int\frac{\mathbf{K}\times\hat{\mathscr{r}}}{\mathscr{r}^2}2\pi R dz$,

It is azimuthal symmetric, so only the z conponent survives dB_z

$\mathscr{r}=R\sec\theta,\ \ z=R\tan\theta,\ dz=R\sec^2\theta d\theta,$

$$dB_z=\frac{\mu_0}{4\pi}\frac{M\sin\theta}{R^2\sec^2\theta}2\pi R\cdot R\sec^2\theta d\theta=\frac{\mu_0}{2}M\sin\theta d\theta$$

$$B_z=\int_{\theta_2}^{\theta_1}\frac{\mu_0}{2}M\sin\theta d\theta=\frac{\mu_0}{2}M(\cos\theta_2-\cos\theta_1)$$